Addendum

1° $\mathbb{M}(V) = \mathbb{N}(V) \oplus \mathbb{N}(V)^* \supset \Lambda(V) = \{ (B_h) \mid \mu = 0, \text{nilpotent} \}$ Lusztig's lagrangian

If Q is of type ADE $\Rightarrow \mathrm{Irr}\,\Lambda(V) \xleftrightarrow{\text{bij}} \mathbb{N}(V)/G_V$ (G_V-orbit)

closure of conormal bundles of orbits

2° ζ : stability condition $\in \mathbb{Q}^{Q_0}$

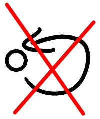
ζ, ζ' generic

(complement of finite hyperplanes) $\Rightarrow D^b \mathrm{Coh}(\mathfrak{M}_\zeta(V, W)) \simeq D^b \mathrm{Coh}(\mathfrak{M}_{\zeta'}(V, W))$

Kaledin 2005

Halpern-Leistner + Sam arXiv:1601.02030

2nd day

I assum 

Recall $\mathfrak{M}(V,W)$, $\mathfrak{M}_0(V,W)$

"semisimple" representation

$\Longrightarrow \quad \pi : \mathfrak{M}(V,W) \longrightarrow \mathfrak{M}_0(V,W)$ projective morphism

quasiprojective variety

affine algebraic variety

Prop $\mathfrak{M}(V,W)$ is a (possibly empty) nonsingular variety of $\dim = 2(\vec{\dim} V, \vec{\dim} W) - (\vec{\dim} V, C\, \vec{\dim} V)$

(This is the expected dimension)

$2I - A$: Cartan matrix

(sketch of the proof)

Consider deformation complex $\underset{\substack{\shortparallel \\ \mathrm{Lie}\, G_V}}{\bigoplus \mathrm{End}\, V_i} \xrightarrow{\text{differential of } G_V\text{-action}} \mathbb{M}(V) \xrightarrow{\text{differential of } \mu} \bigoplus \mathrm{End}\, V_i$

$H^0 = \mathrm{Hom}(X,X) = \mathrm{Lie}\,\mathrm{Aut}(X) \overset{\text{dual by 2-CY}}{\longleftrightarrow} H^2 = \mathrm{Ext}^2(X,X)$ obstruction $\quad H^1 = \mathrm{Ext}^1(X,X)$ deformation

tangent space to $\mathcal{M}(V,W)$

<u>Lemma</u> stability $\Longrightarrow$ $H^0 = 0$ (and hence $H^2 = 0$)

$\because$ $S = \mathrm{Im}\,\xi \quad (\xi \in H^0) \quad \Rightarrow \quad B(S) \subset S,\ J(S) = 0 \quad \therefore S = 0$ i.e., $\xi = 0$ //

<u>Slogan</u> : moduli space is a nonlinear version of $\mathrm{Ext}^\bullet(X,X)$

<u>Remark</u> deformation cpx $\cong D(\qquad)$ $\Rightarrow$ tangent space $\cong D$(tangent space)

symplectic form

Lagrangian subvarieties

$$\mathcal{L}(V,W) := \pi^{-1}(0) \subset \mathfrak{M}(V,W)$$

all maps are $0 \in \mathfrak{M}_0(V,W)$

Prop ① $\mathcal{L}(V,W)$ is a lagrangian subvariety (in particular half dimensional)

② $\mathcal{L}(V,W) \underset{\text{homotopic}}{\simeq} \mathfrak{M}(V,W)$ $\Rightarrow$ $H_{\text{middle}}(\mathfrak{M}(V,W))$ has a base given by $\operatorname{Irr}\mathcal{L}(V,W)$

③ $\mathcal{L}(V,W) = (\Lambda(V) \times \operatorname{Hom}(V,W))^{\text{stable}}$

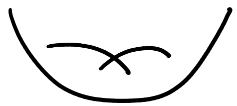
Examples

- A_1 $\quad V \overset{\downarrow\uparrow}{} W$ $\qquad \mathfrak{M}(V,W) = T^*Gr(\dim V, \dim W) \supset \mathcal{L}(V,W) = Gr(\dim V, \dim W)$

- A_2

$$\begin{tikzcd} V_1=\mathbb{C} \arrow[r, shift left] \arrow[d, shift left] & V_2=\mathbb{C} \arrow[l, shift left] \arrow[d, shift left] \\ W_1=\mathbb{C} \arrow[u, shift left] & W_2=\mathbb{C} \arrow[u, shift left] \end{tikzcd} \qquad z,\ x,\ y$$

$\mathfrak{M}_0(V,W) = xy = z^3$ in $\mathbb{C}^3$

$\mathfrak{M}(V,W) =$ its resolution 

$\mathcal{L}(V,W) = \mathbb{P}^1 \cup \mathbb{P}^1$ intersecting a pt

$$\begin{tikzcd} \mathbb{C} \arrow[d] & \mathbb{C} \arrow[l] \arrow[d] \arrow[l, dashed, shift left, "0"] \\ \mathbb{C} & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \arrow[d] \arrow[r, dashed, "0"'] & \mathbb{C} \arrow[d] \arrow[l, shift right] \\ \mathbb{C} & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \arrow[d] \arrow[r, dashed, "0"] & \mathbb{C} \arrow[d] \arrow[l, dashed, "0"] \\ \mathbb{C} & \mathbb{C} \end{tikzcd}$$

Hecke correspondence

☆ Next we "relate" $\mathfrak{M}(V,W)$ for various V.

Choose $i \in Q_0$, and take V, V' such that

$$☆ \quad \dim V'_j = \begin{cases} \dim V_j - 1 & \text{if } j = i \\ \dim V_j & \text{if } j \neq i \end{cases}$$

Consider a pair $(B, I, J) \in \mathfrak{M}(V, W)$ (W is in common)
$(B', I', J') \in \mathfrak{M}(V', W)$

s.t. $\exists\ \xi : V' \longrightarrow V$ intertwining (B, I, J) & (B', I', J')

Thanks to the stability condition, such ξ is unique if exists, and is injective

Thus (B, I, J) is a submodule of (B', I', J').

<u>Remark</u> This construction makes sense if $\dim V_j \geq \dim V'_j \ \forall j$, but ☆ means a "minimum" difference.

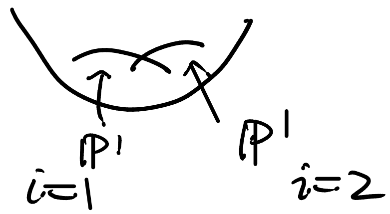
$$\mathfrak{P}_i(V,W) = \{ ((B,I,J),(B',I',J')) \mid \text{as above} \}$$

$$\begin{tikzcd} \mathfrak{P}_i(V,W) \arrow[d, hook] \\ \mathfrak{M}(V',W)\times\mathfrak{M}(V,W) \end{tikzcd}$$

<u>Prop</u> $\mathfrak{P}_i(V,W)$ is a smooth lagrangian subvariety in the product.

<u>Example</u> Recall A_2

$$\begin{tikzcd} V_1=\mathbb{C} \arrow[r, shift left] \arrow[d, shift left] & V_2=\mathbb{C} \arrow[l, shift left] \arrow[d, shift left] \\ W_1=\mathbb{C} \arrow[u, shift left] & W_2=\mathbb{C} \arrow[u, shift left] \end{tikzcd}$$



One check that $\mathfrak{M}(V',W) = pt$ $\quad \overrightarrow{\dim} V' = \overrightarrow{\dim} V - (1,0)$ or $(0,1)$

$\therefore$ $\mathfrak{P}_i(V,W)$ is a subvariety in $\mathfrak{M}(V,W)$.
$(i=1,2)$

It is the $\mathbb{P}^1$ above.

convolution product

Consider (topological) homology group $H_*(\mathfrak{M}(V,W)) \left(\cong H_*(\mathfrak{L}(V,W))\right)$

The Hecke correspondence $\mathfrak{P}_i(V,W)$ defines operators

$$
\begin{tikzcd}
 & \mathfrak{P}_i(V,W) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathfrak{M}(V',W) & & \mathfrak{M}(V,W)
\end{tikzcd}
$$

$$
H_*(\mathfrak{M}(V',W)) \underset{p_{1*}\circ p_2^*}{\overset{p_{2*}\circ p_1^*}{\rightleftarrows}} H_*(\mathfrak{M}(V,W))
$$

(Pull-backs p_1^*, p_2^* are defined via Poincaré duality)

Th ① Operators

$$
e_i = p_{1*}\circ p_2^*, \quad f_i = \pm\, p_{2*}\circ p_1^*, \quad h_i = \left(\dim W_i - \sum_j C_{ij}\dim V_j\right)\mathrm{id}
$$

defines a representation of $\mathfrak{g}_Q$ = Kac-Moody Lie alg. associated with Q

② $\bigoplus_V H_{\mathrm{middle}}(\mathfrak{M}(V,W)) \cong \bigoplus_V H_{\mathrm{top}}(\mathfrak{L}(V,W))$ is preserved, and irreducible integrable highest wt representation

- $H_{top}(\mathfrak{L}(V,W))$ ··· weight sp with weight $\overrightarrow{\dim} W - C\overrightarrow{\dim} V$
- $H_{top}(\underbrace{\mathfrak{L}(0,W)}_{\text{"}pt})\cong \mathbb{C}$: highest weight space

Proof is easy, once the statement is found.

$[e_i, f_j] = \delta_{ij} h_i$ (computation of $\mathrm{Ext}^1(S_i, M)$)

Philosophical Analogy:

$\mathfrak{M}(V,W)$, $\mathfrak{M}(V',W)$ $\longleftrightarrow$ algebras A, A'

homology $\longleftrightarrow$ Grothendieck group of module categories

$\mathfrak{P}_i(V,W) \longleftrightarrow A\otimes A'$-bimodule M

convolution product $\leftrightarrow$ $-\otimes_A M$, $-\otimes_{A'} M$

For Ariki-LLT theory, Lie alg. action is defined in this way.
KLR algebras

"Meta Theorem"

Structures on Lie algebra representations (integrable) are realised by quiver varieties

- Weyl group symmetry $\longleftrightarrow$ reflection functors on quiver varieties
- Tensor product of representations $\longleftrightarrow$ tensor product varieties

$$W = W_1 \oplus W_2 \qquad V = V_1 \oplus V_2$$

stable envelop $\subset \mathfrak{M}(V,W) \times \mathfrak{M}(V_1,W_1) \times \mathfrak{M}(V_2,W_2)$

(Maulik-Okounkov)

- Kashiwara's crystal base $\longleftrightarrow$ Irreducible components of $\mathfrak{L}(V,W)$ and their inductive construction